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Note

Counting acyclic digraphs by sources and sinks

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Abstract

We count labeled acyclic digraphs according to the number sources, sinks, and edges.

1. Counting acyclic digraphs by sources

Let

$$A_n(t;\alpha) = \sum_D \alpha^{s(D)} t^{e(D)},$$

where the sum is over all acyclic digraphs D on the vertex set $[n] = \{1, 2, \ldots, n\}$, $e(D)$ is the number of edges of D, and $s(D)$ is the number of sources of D; that is, the number of vertices of D of indegree 0. Let $A_n(t) = A_n(t;1)$.

To find a recurrence for $A_n(t,\alpha)$, we take an acyclic digraph and add some new vertices as sources. In the digraph we obtain, the new vertices will be a subset of the set of sources.

Lemma 1.

$$\sum_{j=0}^{n} (1+t)^{j(n-j)} \binom{n}{j} \alpha^j A_{n-j}(t) = A_n(t;\alpha+1). \tag{1}$$

Proof. We count triples (S,D,E), where S is a subset of $[n]$, D is an acyclic digraph on $[n]-S$, and E is a subset of the set $S \times ([n]-S)$. We think of the elements of E as edges from S to D. To a triple (S,D,E) we assign the weight $\alpha^j t^e$, where j is the size of S and e is the total number of edges in E and in D. It is clear that the sum of

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the weights of these triples in which $|S| = j$ is $(1+t)^{j(n-j)}\binom{n}{j}\alpha^j A_{n-j}(t)$, and summing on j yields the left hand side of (1).

To a triple (S,D,E) we may associate the pair (S,D') in which D' is the digraph on $[n]$ whose edges are those of D together with the edges in E. Note that S is a subset of the set of sources of D'. It is easily seen that this correspondence gives a bijection from triples (S,D,E) to pairs (S,D') in which D' is an acyclic digraph on $[n]$ and S is a subset of the set of sources of D'. This proves (1). □

In working with recurrences like (1), generating functions of the form

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}$$

are useful, since the convolution

$$\sum_{j=0}^{n} (1+t)^{j(n-j)} \binom{n}{j} a_j b_{n-j} = c_n$$

is equivalent to the generating function equation

$$\left[\sum_{n=0}^{\infty} a_n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}\right] \left[\sum_{n=0}^{\infty} b_n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}\right] = \sum_{n=0}^{\infty} c_n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}.$$

For some further applications of these generating functions, which we call *graphic generating functions*, see [5,6].

We can now express a generating function for $A_n(t;\alpha)$ in terms of the power series

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{(1+t)^{\binom{n}{2}} n!}.$$

Theorem 1.

$$\sum_{n=0}^{\infty} A_n(t;\alpha) \frac{x^n}{(1+t)^{\binom{n}{2}} n!} = F((\alpha-1)x)/F(-x)$$

$$= \left[\sum_{n=0}^{\infty} (\alpha-1)^n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}\right] \Big/ \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}\right]. \tag{2}$$

Proof. Multiplying (1) by $x^n/(1+t)^{\binom{n}{2}} n!$ and summing on n yields

$$F(\alpha x) \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{(1+t)^{\binom{n}{2}} n!} = \sum_{n=0}^{\infty} A_n(t;\alpha+1) \frac{x^n}{(1+t)^{\binom{n}{2}} n!}. \tag{3}$$

Now let us set $\alpha = -1$ in (3). Since every digraph with at least one vertex has a source, $A_n(t;0) = 0$ for $n > 0$. Thus, we obtain

$$F(-x)\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = 1,$$

and so

$$\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = 1/F(-x) = \left[\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(1+t)^{\binom{n}{2}}n!}\right]^{-1}. \tag{4}$$

Replacing α with $\alpha - 1$ in (3), and using (4), yields (2). □

Formula (4) was found (with $t = 1$) by Robinson [9] (see also [11,10]). If we equate coefficients of α^k in (2), we find that the graphic generating function for acyclic digraphs with k sources is

$$\frac{x^k}{(1+t)^{\binom{k}{2}}k!}\;\frac{F\left(-x/(1+t)^k\right)}{F(-x)}$$

as shown by Liskovets [7].

The approach used to derive Lemma 1 can also be applied to other types of acyclic digraphs. For example, in counting acyclic functional digraphs, a new vertex can be added as a source to a digraph with m vertices in m ways. We obtain the following analogue of Lemma 1.

Proposition. *Let R be a set of size r disjoint from the set of positive integers. Let*

$$T_n(\alpha) = \sum_V \alpha^{s(V)},$$

where the sum is over all acyclic functional digraphs V corresponding to functions from $[n]$ to $[n] \cup R$, and $s(V)$ is the number of elements of $[n]$ which are sources of V. (Thus, $T_n(\alpha)$ counts forests of r rooted trees by the number of non-root leaves.) Let $T_n = T_n(1)$. Then

$$\sum_{j=0}^{n}(n-j+r)^j\binom{n}{j}\alpha^j T_{n-j} = T_n(\alpha+1). \tag{5}$$

If we multiply (5) by $x^n/n!$ and sum on n, we get

$$\sum_{n=0}^{\infty} T_n(\alpha+1)\frac{x^n}{n!} = e^{\alpha r x}\sum_{k=0}^{\infty} T_k\frac{(xe^{\alpha x})^k}{k!}.$$

Setting $\alpha = -1$ and using the fact that $T_n(0)$ is 1 for $n = 0$ and 0 for $n > 0$, we get a functional equation that can be solved by Lagrange inversion to get $T_n = r(n+r)^{n-1}$, as is well known; thus

$$\begin{aligned}
T_n(\alpha) &= \sum_{j=0}^{n} \binom{n}{j} (\alpha-1)^j r(n-j+r)^{n-1} \\
&= \sum_{k=0}^{n} \alpha^k \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} r(j+r)^{n-1} \\
&= \sum_{k=0}^{n} \sum_{l=1}^{n} \alpha^k r^l (n-k)! \binom{n}{k} \binom{n-1}{l-1} S(n-l, n-k) \quad \text{for } n \geqslant 1,
\end{aligned}$$

where $S(n,k)$ is the Stirling number of the second kind.

For a closely related approach to counting trees by leaves, see [1, Proposition 1]. The same approach can also be used to count k-trees [8].

2. Counting acyclic digraphs by sources and sinks

We now count acyclic graphs by sources and sinks (vertices of outdegree zero). We use the same idea as before: starting with an acyclic digraph, we add some additional vertices as sources and some additional vertices as sinks. There is a small complication that arises for vertices that are both sources and sinks, but it is easy to deal with.

Let

$$B_n(t;\alpha,\beta,\gamma) = \sum_D \alpha^{s_0(D)} \beta^{s_1(D)} \gamma^{i(D)} t^{e(D)},$$

where the sum is over all acyclic digraphs D on $[n]$; here $s_0(D)$ is the number of sources of D that are not sinks, $s_1(D)$ is the number of sinks of D that are not sources, and $i(D)$ is the number of isolated vertices of D, i.e., vertices that are both sources and sinks.

By the same reasoning as in the proof of Lemma 1, we get a formula for B_n analogous to (1) that yields the following generating function.

Lemma 2.

$$F(\alpha x) \left[\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{(1+t)^{\binom{n}{2}} n!} \right] F(\beta x)$$

$$= \sum_{n=0}^{\infty} B_n(t; \alpha+1, \beta+1, \alpha+\beta+1) \frac{x^n}{(1+t)^{\binom{n}{2}} n!}. \tag{6}$$

From Lemma 2, we derive a generating function for $B_n(t;\alpha,\beta,\gamma)$.

Theorem 2.

$$\sum_{n=0}^{\infty} B_n(t;\alpha,\beta,\gamma)\frac{x^n}{n!} = \mathrm{e}^{(\gamma-\alpha-\beta+1)x}\sum_{n=0}^{\infty} C_n(t;\alpha,\beta)\frac{x^n}{n!}, \tag{7}$$

where

$$\sum_{n=0}^{\infty} C_n(t;\alpha,\beta)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = \frac{F((\alpha-1)x)F((\beta-1)x)}{F(-x)}. \tag{8}$$

Proof. Since any acyclic digraph consists of an acyclic digraph without isolated vertices together with a set of isolated vertices, we have the recurrence

$$B_n(t;\alpha,\beta,\gamma) = \sum_{j=0}^{n}\binom{n}{j}\gamma^j B_{n-j}(t;\alpha,\beta,0),$$

or equivalently,

$$\sum_{n=0}^{\infty} B_n(t;\alpha,\beta,\gamma)\frac{x^n}{n!} = \mathrm{e}^{\gamma x}\sum_{n=0}^{\infty} B_n(t;\alpha,\beta,0)\frac{x^n}{n!}. \tag{9}$$

Now let $C_n(t;\alpha,\beta) = B_n(t;\alpha,\beta,\alpha+\beta-1)$. Then (7) follows from (9), and (8) follows from (6) and (4). $\square$

The polynomials $B_n(t;\alpha,\beta,\gamma)$ are easily computed from (7) and (8). Note that (7) is an exponential generating function, whereas (8) is a graphic generating function.

The first few values of the polynomials $B_n = B_n(t;\alpha,\beta,\gamma)$ are

$$\begin{aligned}
B_0 &= 1,\\
B_1 &= \gamma,\\
B_2 &= \gamma^2 + 2\alpha\beta t,\\
B_3 &= 3\alpha\beta^2t^2 + 3\alpha^2\beta t^2 + 6\alpha\beta t\gamma + \gamma^3 + 6\alpha\beta t^3 + 6\alpha\beta t^2,\\
B_4 &= 12\alpha^2\beta^2t^2 + 24\alpha\beta t^3 + 4\alpha^3\beta t^3 + 36\alpha^2\beta t^3 + 24\alpha^2\beta^2t^3 + 6\alpha^2\beta^2t^4 + 48\alpha^2\beta t^4\\
&\quad +36\alpha\beta^2t^3 + 48\alpha\beta^2t^4 + 84\alpha\beta t^4 + 4\alpha\beta^3t^3 + 84\alpha\beta t^5 + 12\alpha\beta^2t^5 + 12\alpha^2\beta t^5\\
&\quad +24\alpha\beta t^6 + \gamma^4 + 12\alpha\beta t\gamma^2 + 12\alpha\beta^2t^2\gamma + 12\alpha^2\beta t^2\gamma + 24\alpha\beta t^3\gamma + 24\alpha\beta t^2\gamma.
\end{aligned}$$

The polynomials $S_n(t)$ that count acyclic digraphs on $[n]$ with a single source, a single sink, and no isolated vertices are

$$\begin{aligned}
S_2(t) &= 2t,\\
S_3(t) &= 6t^2 + 6t^3,\\
S_4(t) &= 24t^3 + 84t^4 + 84t^5 + 24t^6,\\
S_5(t) &= 120t^4 + 960t^5 + 2660t^6 + 3500t^7 + 2400t^8 + 840t^9 + 120t^{10}\\
S_6(t) &= 720t^5 + 10\,800t^6 + 59\,280t^7 + 170\,250t^8 + 296\,010t^9 + 334\,680t^{10}
\end{aligned}$$

$$+253\,920t^{11} + 129\,300t^{12} + 42\,660t^{13} + 8280t^{14} + 720t^{15},$$

$$S_7(t) = 5040t^6 + 126\,000t^7 + 1\,175\,580t^8 + 5\,931\,240t^9 + 18\,958\,842t^{10}$$
$$+41\,833\,302t^{11} + 67\,063\,080t^{12} + 80\,561\,460t^{13} + 73\,777\,620t^{14}$$
$$+51\,838\,080t^{15} + 27\,842\,220t^{16} + 11\,261\,460t^{17} + 3\,327\,660t^{18}$$
$$+679\,140t^{19} + 85\,680t^{20} + 5040t^{21}.$$

It may be noted that Theorems 1 and 2 have analogues in the context of the Cartier–Foata theory of free partially commutative monoids [3]: the analogue of (4) is due to Cartier and Foata [3, Théorème 2.4], the analogue of Theorem 1 is due in the case of a single source to Foata [4, Theorem 3.1] and in the general case to Viennot [12, Proposition 5.3], and the analogue of Theorem 2 is due to Bousquet–Mélou [2, Lemme 1.2 and Théorème 1.3].

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